Counting Independent Sets Using the Bethe Approximation

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Abstract

We consider the #P complete problem of counting the number of independent sets in a given graph. Our interest is in understanding the effectiveness of the popular Belief Propagation (BP) heuristic. BP is a simple and iterative algorithm that is known to have at least one fixed point. Each fixed point corresponds to a stationary point of the Bethe free energy (introduced by Yedidia, Freeman and Weiss (2004) in recognition of Hans Bethe's earlier work (1935)). The evaluation of the Bethe Free Energy at such a stationary point (or BP fixed point) leads to the Bethe approximation to the number of independent sets of the given graph. In general BP is not known to converge nor is an efficient, convergent procedure for finding stationary points of the Bethe free energy known. Further, effectiveness of Bethe approximation is not well understood.

As the first result of this paper, we propose a BP-like algorithm that always converges to a BP fixed point for *any* graph. Further, it finds an ε approximate fixed point in $O(n^2 d^4 2^d \varepsilon^{-4} \log^3(n\varepsilon^{-1}))$ iterations for a graph of n nodes with max-degree d. As the next step, we study the quality of this approximation. Using the recently developed 'loop series' approach by Chertkov and Chernyak, we establish that for any graph of n nodes with max-degree d and girth larger than $8d \log_2 n$, the multiplicative error decays as $1 + O(n^{-\gamma})$ for some $\gamma > 0$. This provides a deterministic counting algorithm that leads to strictly different results compared to a recent result of Weitz (2006).

Finally as a consequence of our results, we prove that the Bethe approximation is exceedingly good for a random 3-regular graph conditioned on the Shotest Cycle Cover Conjecture of Alon and Tarsi (1985) being true.

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